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# ON POSITIVITY PRESERVING FINITE VOLUME SCHEMES FOR COMPRESSIBLE EULER EQUATIONS

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## ABSTRACT

We consider positivity preserving property of first and higher order finite volume schemes for one and two dimensional compressible Euler equations of gas dynamics. A general framework is established which shows the positivity of density and pressure whenever the underlying one dimensional first order building block based on an exact or approximate Riemann solver and the reconstruction are both positivity preserving. Appropriate limitation to achieve high order positivity preserving reconstruction is described.

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# 1 Introduction

To solve a scalar conservation law

$$u_t + \operatorname{div} f(u) = 0 \tag{1.1}$$

with possibly discontinuous solutions, one usually studies the total variation stability (in one space dimension) or  $L^\infty$  stability (in multi space dimensions) of the numerical schemes. The schemes can usually be generalized to systems via local characteristic decompositions and usually work equally well numerically. However, no stability property can be automatically carried over to the nonlinear system case. For example, most second order TVD schemes, or even some first order monotone schemes, when generalized to compressible Euler equations of gas dynamics, do not always preserve the positivity of density and pressure. This may cause problems in practical calculations when the solution is near vacuum, for example in the computation of blast waves.

When considering the compressible Euler equations of gas dynamics, a natural stability condition for the numerical approximation is positivity preserving for density and pressure. Another possible à priori bound is the maximum principle for the specific entropy (Tadmor [10]), which seems extremely difficult to preserve for second and higher order schemes (Khobalatte and Perthame [5]).

The positivity of density and pressure is already interesting because it allows one to derive a rigorous CFL condition (limitation of the time step to use). Since we consider conservative schemes, it also allows one to obtain à priori  $L^1$  bounds on the density, momentum and energy. In this sense, positivity of density and pressure is a *weak* stability condition. Of course for these nonlinear systems, such a weak stability is not enough to assert the convergence of the method and estimates on the derivatives are usually needed for that purpose. Note however that for a large class of  $2 \times 2$  systems in one dimension, weak stability (say in  $L^\infty$ ) is enough to prove the convergence of the method.

In this paper we will provide a general framework and illustrate by several examples the

way to impose positivity of density and pressure for finite volume schemes, for one and two space dimensions and for first and higher order accuracy. Of course the first ingredient is a positivity preserving exact or approximate Riemann solver, such as Godunov, Lax-Friedrichs, Boltzmann type (Perthame [6]), and Harten-Lax-van Leer [3]. For a first order scheme, this is enough to guarantee positivity of density and pressure, even for two dimensional problems set on an arbitrary triangulation. However, for higher order finite volume schemes, the reconstruction must also satisfy such a property. We will show that only the nodal values needed for constructing the flux along a cell edge must be positive, in order to obtain a positive scheme. The reconstructed function, which is a piecewise polynomial and can be obtained in ENO spirit, needs not be positive everywhere.

This paper is organized as follows: we first prove a positivity result and apply it to a third order scheme in one space dimension in Section 2. Then, in Section 3, we consider first order positive schemes in two space dimensions for arbitrary triangulations. The fourth section is devoted to second order schemes in two space dimensions. In the Appendix we recall why Lax-Friedrichs scheme is positivity preserving.

## 2 Positivity in One Space Dimension and a Positive Third Order Scheme

In this section we consider the one dimensional compressible Euler equations for perfect gas:

$$U_t + F(U)_x = 0, \quad t \geq 0, x \in R \quad (2.1)$$

with

$$U = (\rho, \rho u, E), \quad E = \frac{1}{2} \rho u^2 + \rho e \quad (2.2)$$

$$F(U) = (\rho u, \rho u^2 + p, (E + p)u), \quad p = (\gamma - 1) \rho e \quad (2.3)$$

where  $\rho$  is the density,  $u$  is the velocity,  $E$  is the total energy,  $p$  is the pressure,  $e$  is the internal energy, and  $\gamma > 1$  is a constant ( $\gamma = 1.4$  for air).

Let us recall for later purpose that for the pressure law in (2.3), the speed of sound is given by  $c = \sqrt{\gamma(\gamma - 1)e}$  and thus the three eigenvalues of the system are  $u, u \pm c$ .

We give conditions on numerical schemes in order to satisfy the positivity property, by which we mean that the resulting value of  $U$  should satisfy  $\rho > 0$  ( $\rho_j^n > 0$  on the discrete level) and  $p > 0$  ( $p_j^n > 0$  on the discrete level), if the initial condition satisfies those positivity conditions.

We first present the finite volume schemes under consideration, then we give a general positivity theorem, and finally we show how a third order reconstruction can be modified in order to satisfy the assumption of the general theorem.

A general finite volume scheme can be written as

$$\bar{U}_j^{n+1} = \bar{U}_j^n - \lambda \left[ h \left( U_{j+\frac{1}{2}}^+, U_{j+\frac{1}{2}}^- \right) - h \left( U_{j-\frac{1}{2}}^+, U_{j-\frac{1}{2}}^- \right) \right] \quad (2.4)$$

where  $n \geq 0$  refers to the time step  $\Delta t$  and  $j \in Z$  to the uniform space discretization of the size  $\Delta x$  (to simplify the presentation), and  $\lambda = \frac{\Delta t}{\Delta x}$ .  $\bar{U}_j^n$  are approximations to the cell averages of  $U$  in the cell  $C_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$  at time level  $n$ , and  $U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+$  are high order approximations of the nodal values  $U^n(x_{j+\frac{1}{2}})$  within the cells  $C_i$  and  $C_{i+1}$ , respectively. These values can either be evolved as independent degrees of freedom such as in the discontinuous Galerkin finite element method (see, e.g., [1]), or be reconstructed from the cell averages  $\bar{U}_j^n$ . Let us recall that the general ENO philosophy [4] allows us to reconstruct, from the cell averages  $\bar{U}_j^n$ , a piecewise polynomial function  $U^n(x)$  which is high order accurate ( $r$ -th order if  $U^n(x)$  is piecewise  $(r - 1)$ -th order polynomials), and conserves the local mean:

$$\frac{1}{\Delta x} \int_{C_j} U^n(x) dx = \bar{U}_j^n \quad (2.5)$$

The nodal values needed in scheme (2.4) can then be set to

$$U_{j+\frac{1}{2}}^+ = U^n \left( x_{j+\frac{1}{2}}^+ \right), \quad U_{j+\frac{1}{2}}^- = U^n \left( x_{j+\frac{1}{2}}^- \right) \quad (2.6)$$

since the function  $U^n(x)$  is discontinuous at the cell interface  $x_{j+\frac{1}{2}}$ .

What remains to be explained for scheme (2.4) is the flux function  $h(U, V)$ . This is assumed to be an exact or, in most cases, an approximate Riemann solver. In particular,  $h(U, V)$  is a Lipschitz continuous function of both arguments, and is consistent with the physical flux  $F(U)$  in (2.3):  $h(U, U) = F(U)$ . We will further make the following

**Assumption 1:**  $h(\cdot, \cdot)$  produces a one dimensional first order scheme which satisfies the positivity property under a CFL condition:

$$\lambda \|(|u| + c)\|_\infty \leq \alpha_0 \quad (2.7)$$

i.e., if  $\rho_j^n > 0$  and  $p_j^n > 0$  for all  $j \in Z$ , then the solution  $U_j^{n+1}$  of

$$U_j^{n+1} = U_j^n - \lambda [h(U_{j+1}^n, U_j^n) - h(U_j^n, U_{j-1}^n)] \quad (2.8)$$

also satisfies  $\rho_j^{n+1} > 0$  and  $p_j^{n+1} > 0$  under the CFL condition (2.7).

We will call such a  $h(\cdot, \cdot)$  positivity preserving under the CFL condition (2.7). Examples of positivity preserving (approximate) Riemann solvers include Godunov, Lax-Friedrichs (see the Appendix), Boltzmann type [6] and Harten-Lax-van Leer [3].

Our first result is

**Theorem 1.** Assume  $h(\cdot, \cdot)$  satisfies Assumption 1. If the reconstructed nodal values  $U_{j+\frac{1}{2}}^\pm$  have positive density and pressure for all  $j \in Z$ , then the full scheme (2.4) is positivity preserving under the CFL condition

$$\lambda \|(|u| + c)\|_\infty \leq \alpha \alpha_0 \quad (2.9)$$

where  $0 < \alpha \leq 1$  is sufficiently small such that

$$\overline{U}_j^n - \alpha(U_{j+\frac{1}{2}}^- + U_{j-\frac{1}{2}}^+) := V_j \quad (2.10)$$

has positive density and pressure for all  $j \in Z$ .

**Remark 1.** Of course  $\alpha = 0$  works. However, since  $U_{j\mp\frac{1}{2}}^\pm$  are close to  $\overline{U}_j^n$ , we can expect that  $\alpha$  is close to  $\frac{1}{2}$ .



**Remark 2.** Here we restrict ourselves to first order Euler forward in time. TVD type high order Runge-Kutta time discretization (Shu and Osher [9]) will keep the validity of Theorem 1, as it was shown in [5].

**Proof of Theorem 1:** Let us introduce three “first order” schemes within the cell  $C_j$ :

$$\begin{aligned}\tilde{U}_j^+ &= U_{j+\frac{1}{2}}^- - \frac{\lambda}{\alpha} \left[ h \left( U_{j+\frac{1}{2}}^+, U_{j+\frac{1}{2}}^- \right) - h \left( U_{j+\frac{1}{2}}^-, V_j \right) \right] \\ \tilde{V}_j &= V_j - \lambda \left[ h \left( U_{j+\frac{1}{2}}^-, V_j \right) - h \left( V_j, U_{j-\frac{1}{2}}^+ \right) \right] \\ \tilde{U}_j^- &= U_{j-\frac{1}{2}}^+ - \frac{\lambda}{\alpha} \left[ h \left( V_j, U_{j-\frac{1}{2}}^+ \right) - h \left( U_{j-\frac{1}{2}}^+, U_{j-\frac{1}{2}}^- \right) \right]\end{aligned}\tag{2.11}$$

The three of them are of the type (2.8), with possibly  $\frac{\lambda}{\alpha}$  in the place of  $\lambda$ . Therefore under the CFL condition (2.9), the values  $\tilde{U}^+$ ,  $\tilde{U}^-$  and  $\tilde{V}$  all have positive density and pressure. We set

$$\overline{U}_j^{n+1} = \alpha(\tilde{U}_j^+ + \tilde{U}_j^-) + \tilde{V}_j\tag{2.12}$$

Then the linear combination of the above three equalities in (2.11) with weights  $(\alpha, 1, \alpha)$  just gives the scheme (2.4), thanks to the definition of  $V_j$  in (2.10).

Finally, (2.12) implies that, by concavity,  $\overline{U}_j^{n+1}$  has also positive density and pressure. The pressure is indeed a concave function of  $(\rho, \rho u, E)$ .

□

Theorem 1 tells us that, if we use a positivity preserving approximate Riemann solver, then we only need to worry about positive density and pressure in the nodal values  $U_{j+\frac{1}{2}}^\pm$ , either from the reconstruction or from direct evolution. This can usually be achieved by a further limitation upon  $U_{j+\frac{1}{2}}^\pm$ , such that positivity of density and pressure is enforced and accuracy is preserved in smooth regions. We now show, as an example, how this can be done for a third order reconstruction. Let us consider a typical cell  $(-\delta, \delta)$  where  $\delta = \frac{\Delta x}{2}$ , in which we are given three cell averages  $(\bar{\rho}, \bar{\rho u}, \bar{E})$  with positive density and pressure. Let us

build three quadratic polynomials

$$\begin{aligned}\rho(x) &= \rho_0 + \rho_1 x + \rho_2 \frac{x^2}{2} \\ u(x) &= u_0 + u_1 x + u_2 \frac{x^2}{2} \\ p(x) &= p_0 + p_1 x + p_2 \frac{x^2}{2}\end{aligned}\tag{2.13}$$

obtained by (i) fitting the average of  $(\rho, \rho u, \frac{\rho u^2}{2} + \frac{p}{\gamma-1})$  to  $(\bar{\rho}, \bar{\rho u}, \bar{E})$  and (ii) fitting the nodal values  $(\rho, \rho u, \frac{\rho u^2}{2} + \frac{p}{\gamma-1})(\pm\delta)$  with the values obtained by a third order reconstruction, say from the ENO procedure. These might be associated to a negative density and pressure. Then we need an additional limitation. It can always be performed easily as shown in

**Proposition 2:** Given the three quadratic polynomials in (2.13), we can always perform a limitation of their coefficients so that (i) the means of the three conserved variables are preserved, and (ii) the nodal values have positive density and pressure.

**Remark:** The precise limitation, and the deduction of the appropriate coefficients in (2.13) are given in the proof below.

Of course, combining Theorem 1 and Proposition 2, we obtain a positive preserving scheme. A similar construction for second order schemes can be found in [6] and [5].

**Proof of Proposition 2:** First of all, let us fix some notations. Being given the cell averages  $(\bar{\rho}, \bar{\rho u}, \bar{E})$ , we also define  $\bar{u}$  and  $\bar{p}$  through

$$\bar{\rho u} = \bar{\rho} \bar{u}, \quad \bar{E} = \frac{\bar{\rho} \bar{u}^2}{2} + \frac{\bar{p}}{\gamma-1}\tag{2.14}$$

Notice that we are given  $\bar{\rho} > 0$  and  $\bar{p} > 0$ .

First step: The limitation on the density is rather easy. Since  $\bar{\rho} = \rho_0 + \rho_2 \frac{\delta^2}{6}$ , we can modify  $\rho_1$  without altering the conservation of mass. To enforce positivity of  $\rho(\pm\delta)$  is equivalent to impose a simple limitation on  $\rho_1$ :

$$|\rho_1| \delta < \rho_0 + \rho_2 \frac{\delta^2}{2}.\tag{2.15}$$

But the right-hand-side of (2.15) could be negative. Therefore we first impose, for instance,

$$\frac{5\bar{\rho}}{4} > \rho_0 > \frac{3\bar{\rho}}{4}, \quad \bar{\rho} = \rho_0 + \rho_2 \frac{\delta^2}{6} \quad (2.16)$$

Indeed, this implies that

$$|\rho_2| \frac{\delta^2}{2} = 3|\rho_2| \frac{\delta^2}{6} = 3|\bar{\rho} - \rho_0| < \frac{3\bar{\rho}}{4} < \rho_0$$

and now (2.15) can be imposed too.

Second step: Next the conservation of momentum implies the following relation which gives  $u_0$  in terms of  $u_1, u_2$  and  $\rho(x)$ :

$$\bar{\rho}u_0 = \bar{\rho}\bar{u} - \rho_1 u_1 \frac{\delta^2}{3} - u_2 \left( \rho_0 \frac{\delta^2}{6} + \rho_2 \frac{\delta^4}{20} \right) \quad (2.17)$$

And the conservation of energy gives, after some easy calculations

$$\bar{p} = (\gamma - 1) \left[ -\frac{1}{2}\bar{\rho}(u_0 - \bar{u})^2 + u_1^2 \left( \rho_0 \frac{\delta^2}{3} + \rho_2 \frac{\delta^4}{10} \right) + u_1 u_2 \rho_1 \frac{\delta^4}{5} + \rho_2 u_2^2 \frac{\delta^6}{56} \right] + p_0 + p_2 \frac{\delta^2}{6} \quad (2.18)$$

As in the density case, we will impose, in order to get a positive pressure at the nodes, a simple limitation on  $p_1$

$$|p_1|\delta < p_0 + p_2 \frac{\delta^2}{2} \quad (2.19)$$

and thus we need some limitations in order to guarantee that the right-hand-side of (2.19) is positive. As before we first impose the limitations on  $p_0, u_1, u_2$ :

$$p_0 < \frac{5}{4}\bar{p} \quad (2.20)$$

$$u_1^2(\gamma - 1) \left( \rho_0 \frac{\delta^2}{3} + |\rho_1| \frac{\delta^3}{10} + \rho_2 \frac{\delta^4}{10} \right) < \frac{\bar{p}}{12} \quad (2.21)$$

$$u_2^2(\gamma - 1) \left( |\rho_1| \frac{\delta^5}{10} + \rho_2 \frac{\delta^6}{56} \right) < \frac{\bar{p}}{12} \quad (2.22)$$

This defines a unique  $u_0$  through (2.17) and a unique  $p_2$  through (2.18). For these values we deduce from (2.18), using (2.20):

$$\begin{aligned} \frac{\bar{p}}{6} &< \bar{p} - \frac{2}{3}p_0 \\ &= (\gamma - 1) \left[ -\frac{1}{2}\bar{\rho}(u_0 - \bar{u})^2 + u_1^2 \left( \rho_0 \frac{\delta^2}{3} + \rho_2 \frac{\delta^4}{10} \right) + \rho_1 u_1 u_2 \frac{\delta^4}{5} + \rho_2 u_2^2 \frac{\delta^6}{56} \right] + \frac{1}{3} \left( p_0 + p_2 \frac{\delta^2}{2} \right) \\ &\leq (\gamma - 1) \left[ u_1^2 \left( \rho_0 \frac{\delta^2}{3} + \rho_2 \frac{\delta^4}{10} + |\rho_1| \frac{\delta^3}{10} \right) + u_2^2 \left( \rho_2 \frac{\delta^6}{56} + |\rho_1| \frac{\delta^5}{10} \right) \right] + \frac{1}{3} \left( p_0 + p_2 \frac{\delta^2}{2} \right) \end{aligned}$$

and from (2.21)-(2.22) we deduce that  $p_0 + p_2 \frac{\delta^2}{2} > 0$  and thus (2.19) can also be imposed.

This concludes the description of a possible set of limitations, which proves Proposition 2. Notice that these limitations (2.15), (2.16), (2.19), (2.20), (2.21) and (2.22) just amount to avoid gradients or second order derivatives of the order  $\frac{1}{\Delta x}$  (or larger) and thus they preserve the accuracy in smooth regions.

### 3 First Order Positive Schemes in Two Dimensions

We now consider the equations

$$U_t + \operatorname{div} F(U) = 0 \quad t \geq 0, x \in R^2 \quad (3.1)$$

where

$$U = (\rho, \rho u, E), \quad u \in R^2, \quad E = \frac{1}{2} \rho |u|^2 + \rho e \quad (3.2)$$

$$F(U) = (\rho u, \rho u \otimes u + pI, (E + p)u), \quad p = (\gamma - 1)\rho e \quad (3.3)$$

We consider finite volume schemes set on a triangulation  $\mathcal{C}$ . The control volumes will be the triangles  $K \in \mathcal{C}$ . For each triangle  $K$  we denote by  $(l_K^\alpha)_{1 \leq \alpha \leq 3}$  the length of its three edges  $(e_K^\alpha)_{1 \leq \alpha \leq 3}$ , with outward unit normal  $(\nu_K^\alpha)_{1 \leq \alpha \leq 3}$ . Finally,  $K(\alpha)$  will denote the neighboring triangle along  $e_K^\alpha$  and  $|K|$  the area of the triangle  $K$ . Then, we consider the scheme

$$U_K^{n+1} = U_K^n - \frac{\Delta t}{|K|} \sum_{\alpha=1}^3 h(U_{K(\alpha)}^n, U_K^n, \nu_K^\alpha) l_K^\alpha \quad (3.4)$$

for some (approximate) Riemann solver  $h(U, V, \nu)$  in the direction  $\nu$ . We recall some classical examples in the Appendix. The basic properties of  $h$  are now

$$h(U, V, \nu) = -h(V, U, -\nu), \quad (\text{conservativity}) \quad (3.5)$$

$$h(U, U, \nu) = F(U) \cdot \nu \quad (\text{consistency}) \quad (3.6)$$

Next, we also impose a positivity condition for the one dimensional solver obtained by fixing  $\nu$  (see the Appendix for examples). This is the same as Assumption 1 in the previous section, which now means that the solution  $\tilde{U}$ , a four component vector, of

$$\tilde{U} = U - \lambda [h(V, U, \nu) - h(U, W, \nu)] \quad (3.7)$$

has positive density and pressure as soon as  $U, V, W$  do. Now,  $h$  is a four component vector.

The main result of this section is

**Theorem 3:** Let  $h(\cdot, \cdot, \cdot)$  satisfy (3.5)-(3.6) and the one dimensional positivity property Assumption 1. Then the scheme (3.4) satisfies the positivity property under the CFL condition

$$\sum_{\alpha=1}^3 l_K^\alpha \Delta t (|u_K| + c_K) \leq \alpha_0 |K|, \quad \text{for all } K \in \mathcal{C} \quad (3.8)$$

**Proof of Theorem 3:** We define

$$\tilde{U}^\alpha = \frac{l_K^\alpha}{\sum_{\beta=1}^3 |l_K^\beta|} U_K^n - \frac{\Delta t}{|K|} l_K^\alpha \left[ h(U_{K(\alpha)}^n, U_K^n, \nu_K^\alpha) - h(U_K^n, U_K^n, \nu_K^\alpha) \right] \quad (3.9)$$

Since

$$\sum_{\alpha=1}^3 l_K^\alpha h(U_K^n, U_K^n, \nu_K^\alpha) = F(U_K^n) \cdot \sum_{\alpha=1}^3 \nu_K^\alpha l_K^\alpha = 0$$

we have  $U_K^{n+1} = \sum_{\alpha=1}^3 \tilde{U}^\alpha$ . And we conclude as in the one dimensional case.

## 4 Second Order Positive Schemes in Two Dimensions

We extend the result of the previous section to second order schemes, i.e., in which edge averages of the flux are approximated with second order accuracy. It has to be noted that, as usual, this means that the resulting scheme formally only has a first order truncation error. In the same way the first order schemes used in Section 3 formally only have zeroth order truncation error but it is well known now that they are convergent (see for example the preprints of Szepessy entitled, “Measure valued solutions to conservation laws with boundary conditions”, of Champier, Gallouet and Herbin entitled “Convergence of an upstream finite volume scheme on a triangular mesh for nonlinear hyperbolic equations”, and of Coquel and LeFloch entitled “The finite volume method on general triangulations converges to general conservation laws”).

As for the one dimensional case, we give a general result assuming a positive reconstruction.

We now consider an approximation of the two dimensional Euler equation (3.1) under the form

$$U_K^{n+1} = U_K^n - \frac{\Delta t}{|K|} \sum_{\alpha=1}^3 h(U_{K(\alpha)}^{n,\alpha}, U_K^{n,\alpha}, \nu_K^\alpha) l_K^\alpha \quad (4.1)$$

where we use the notations of Section 3. We still assume that the approximate Riemann solver  $h$  satisfies the one dimensional positivity property Assumption 1. The main difference with the situation in Section 3, is that we now use second order approximations  $U_K^{n,\alpha}$ ,  $U_{K(\alpha)}^{n,\alpha}$  of  $U^n$  at the center of the edge  $e_K^\alpha$ .  $U_K^{n,\alpha}$  is such an approximation in the triangle  $K$ ,  $U_{K(\alpha)}^{n,\alpha}$  is in the triangle  $K(\alpha)$  neighboring  $K$  along the edge  $e_K^\alpha$ . These values can be obtained using a slope reconstruction, or an interpolation together with the interpretation of functions as piecewise constant in subcells as in Perthame and Qiu [7], or being evolved as independent degrees of freedom in the discontinuous Galerkin finite element method [2]. In any case, we can assume that these values are conservative, i.e. that there exists real numbers  $(\omega_K^\alpha)_{1 \leq \alpha \leq 3}$ , such that

$$\omega_K^1 + \omega_K^2 + \omega_K^3 = |K|, \quad \omega_K^\alpha \geq 0; \quad \sum_{\alpha=1}^3 \omega_K^\alpha U_K^{n,\alpha} = |K| U_K^n \quad (4.2)$$

For instance, when  $U_K^{n,\alpha}$  is the value, in the middle of the edge  $e_K^\alpha$ , of a linear function in  $K$ , the coefficients  $\omega_K^\alpha$  are just  $|K|$  times the barycentric coordinates of the mass center of  $K$ , with respect to the middle of the edges. Another example can be found in [7].

In order to state our result, let us introduce some notations: Denoting by  $C_K^\alpha$  the middle point of the edge  $e_K^\alpha$ , we define

$$Y_k = \sum_{\alpha=1}^3 \frac{\omega_K^\alpha C_K^\alpha}{|K|} \quad (4.3)$$

then, the triangle  $K$  is naturally divided into three subtriangles  $(K_\alpha)_{1 \leq \alpha \leq 3}$  obtained by

joining  $Y_K$  to the three vertices of  $K$ .

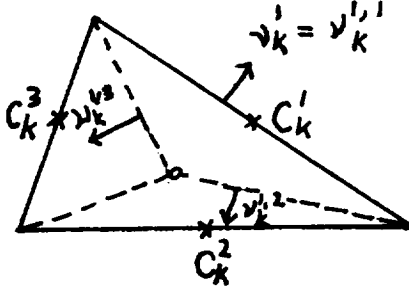


Fig. 1 Sub-triangles decomposition of a triangle  $K$

The unit outward normals of the sub-triangle  $K_\alpha$  are denoted by  $(\nu_K^{\alpha,\alpha'})_{1 \leq \alpha' \leq 3}$  where  $\nu_K^{\alpha,\alpha} = \nu_K^\alpha$ ; and the length of the edges  $(e_K^{\alpha,\alpha'})_{1 \leq \alpha' \leq 3}$ , where  $e_K^{\alpha,\alpha} = e_K^\alpha$ , of the sub-triangle  $K_\alpha$ , are respectively  $(l_K^{\alpha,\alpha'})_{1 \leq \alpha' \leq 3}$  with  $l_K^{\alpha,\alpha} = l_K^\alpha$ .

We can now state our main result:

**Theorem 4:** Let  $h$  satisfy the one dimensional positivity property Assumption 1, and assume (4.2). Then the scheme (4.1) satisfies the positivity property under the CFL condition

$$\sum_{\alpha'=1}^3 l_K^{\alpha,\alpha'} \Delta t (|u_K^\alpha| + c_K^\alpha) \leq \alpha_0 \omega_K^\alpha \quad (4.4)$$

**Proof of Theorem 4:** Mimicking the proof of Theorem 3, let us define  $U_K^{n+1,\alpha}$  by

$$U_K^{n+1,\alpha} \omega_K^\alpha = U_K^{n,\alpha} \omega_K^\alpha - \Delta t \sum_{\beta=1}^3 h(U_K^{n,\alpha,\beta}, U_K^{n,\alpha}, \nu_K^{\alpha,\beta}) l_K^{\alpha,\beta} \quad (4.5)$$

where

$$U_K^{n,\alpha,\alpha} = U_K^{n,\alpha}, \quad U_K^{n,\alpha,\beta} = U_K^{n,\beta} \quad \text{for } \beta \neq \alpha \quad (4.6)$$

Adding the equalities (4.5) for  $\alpha = 1, 2, 3$  we obtain (4.1) with

$$U_K^{n+1} = \sum_{\alpha=1}^3 U_K^{n+1,\alpha} \omega_K^\alpha \quad (4.7)$$

Indeed, this is just a consequence of (4.2) and, for  $\alpha \neq \beta$ ,

$$h(U_K^{n,\alpha,\beta}, U_K^{n,\alpha}, \nu_K^{\alpha,\beta}) = -h(U_K^{n,\beta,\alpha}, U_K^{n,\beta}, \nu_K^{\beta,\alpha}) \quad (4.8)$$

$$l_K^{\alpha,\beta} = l_K^{\beta,\alpha} \quad (4.9)$$

Now, applying Theorem 3 to (4.5), we obtain that the  $U_K^{n+1,\alpha}$  satisfy the positivity property. By concave combination  $U_K^{n+1}$  satisfies it too and the theorem is proved.

□

Notice that (4.2) is not essential. It allows us to simplify greatly the statement of Theorem 4. If it is not satisfied, we require a two dimensional extension of the assumption (2.10) for the one dimensional case which could be quite difficult to check here. Finally, we refer to [7] for an example of a reconstruction which satisfies the positivity property, together with (4.2). There the control volume is a dual mesh and the positivity is proved using Boltzmann solvers. Our approach here could be extended to a dual mesh and the full scheme, using a Lax-Friedrichs or Godunov solver, satisfies also the positivity property.

## 5 Conclusion

We have considered how to preserve the positivity of density and pressure for solving compressible Euler equations using finite volume numerical methods. A general framework is established to obtain positivity preserving first and higher order schemes for one and two space dimensions.



## 6 Appendix: Positivity of the Lax-Friedrichs Scheme

We consider in this Appendix, the Lax-Friedrichs scheme for the two dimensional situation in Section 3. Fixing a unit vector  $\nu$ , the velocity vector can be written as  $u = (v, w)$  in the frame  $(\nu, \nu^\perp)$ . Then, we define

$$U = (\rho, \rho v, \rho w, E), \quad E = \frac{1}{2}\rho(v^2 + w^2) + \rho e, \quad (6.1)$$

$$H(U) = F(U) \cdot \nu = (\rho v, \rho v^2 + p, \rho v w, (E + p)v) \quad (6.2)$$

The Lax-Friedrichs flux is

$$h(V, U) = \frac{1}{2}(H(U) + H(V) - \beta(V - U)) \quad (6.3)$$

with

$$\beta = \|(|u| + c)\|_\infty \quad (6.4)$$

Let us consider  $U_j^n, U_{j\pm 1}^n$  with positive density and pressure, and set

$$U_j^{n+1} = U_j^n - \lambda [h(U_{j+1}^n, U_j^n) - h(U_j^n, U_{j-1}^n)] \quad (6.5)$$

Then  $U_j^{n+1}$  also has positive density and pressure. An easy way to see that (Sanders [8]) consists of introducing a splitting of the equation

$$U_t + H(U)_x = 0 \quad (6.6)$$

by

$$U_t + (H(U) + \beta U)_x = 0 \quad (6.7)$$

$$U_t + (H(U) - \beta U)_x = 0 \quad (6.8)$$

In other words, we write an exact solver for (5.7)-(5.8), which is easy because there are no longer wave interactions with the choice of  $\beta$  in (6.4):

$$\tilde{U} = U_j - \lambda [H(U_j^n) + \beta U_j^n - H(U_{j-1}^n) - \beta U_{j-1}^n] \quad (6.9)$$

$$\tilde{\tilde{U}} = U_j - \lambda [H(U_{j+1}^n) - \beta U_{j+1}^n - H(U_j^n) + \beta U_j^n] \quad (6.10)$$

Now  $U_j^{n+1} = \frac{\tilde{U} + \tilde{\tilde{U}}}{2}$  is indeed the solution to the Lax-Friedrichs scheme (6.5), and since the exact solver preserves positivity (notice that the addition of  $\pm\beta U$  is unessential for positivity by Galilean invariance), by concave combination  $U_i^{n+1}$  satisfies also the positivity property. We need the CFL condition

$$\lambda \|(|u| + c)\|_\infty \leq \frac{1}{2} \quad (6.11)$$

in order to avoid the interaction of waves when solving (5.7)-(5.8) by (5.9)-(5.10).

Another version of Lax-Friedrichs scheme consists of introducing exact solvers on a staggered grid. It will also obviously satisfy the positivity property. Both of these Lax-Friedrichs schemes in addition satisfy the maximum principle on the specific entropy (Tadmor [10], [5]).

We would like to conclude this Appendix by a remark raised in [7] on two dimensional schemes. Usually two dimensional schemes have to be written under the form (3.4) where the approximate solver  $h(U, V, \nu)$  is indeed a function of three parameters  $U, V$  and  $\nu$ , satisfying  $h(U, U, \nu) = F(U) \cdot \nu$ . In [7], the notion of “genuinely multidimensional solver” is introduced, where the approximate solver is indeed under the special form  $h(U, V) \cdot \nu$  satisfying  $h(U, U) = F(U)$ . The Lax-Friedrichs scheme is obviously not “genuinely multidimensional” because its value really depends on the frame  $(\nu, \nu^\perp)$  used to write the four components system (6.1)-(6.2).

The only example we know of a “genuinely multidimensional” solver is a particular class of Boltzmann solvers introduced in [7].

## References

- [1] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework*, Math. Comput., v52 (1989), pp.411-435.
- [2] B. Cockburn, S. Hou and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case*, Math. Comput., v54 (1990), pp.545-581.
- [3] A. Harten, P. Lax and B. van Leer, *On upstream differencing and Godunov type schemes for hyperbolic conservation laws*, SIAM Review, v25 (1983), pp.35-61.
- [4] A. Harten, B. Engquist, S. Osher and S. Chakravarthy, *Uniformly high order essentially non-oscillatory schemes, III*, J. Comput. Phys., v71, (1987), pp.231-303.
- [5] B. Khobalatte and B. Perthame, *Maximum principle on the entropy and minimal limitations for kinetic schemes*, Math. Comput., to appear.
- [6] B. Perthame, *Second order Boltzmann schemes for compressible Euler equations in one and two space variables*, SIAM J. Numer. Anal., v29 (1992), pp.1-19.
- [7] B. Perthame and Y. Qiu, *A variant of van Leer's method for multidimensional systems of conservation laws*, INRIA Report No. 1562 (1991).
- [8] R. Sanders, private communication.
- [9] C.-W. Shu and S. Osher, *Efficient implementation of essentially non-oscillatory shock capturing scheme*, J. Comput. Phys., v77 (1988), pp.439-471.
- [10] E. Tadmor, *A minimum entropy principle in the gas dynamics equations*, Appl. Num. Math., v2 (1986), pp.211-219.



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